combine even when the forces acting are due to forces of the type e^2/r^2, instead of

$$\frac{(D-1)e^2}{2\pi N r^5}$$

which are much weaker.

[1] Loeb and Ashley, *Proc. Nat. Acad. Sci.*, **10**, 351 (1924).
[2] Loeb, L. B., *Ibid.*, **12**, 35 and 42 (1926).
[3] Tyndall and Phillips, *Proc. Roy. Soc.*, **A111**, 577 (1926).
[4] Mayer, H., *Physik. Zeits.*, **27**, 513 (1926).
[5] Loeb, L. B., *Proc. Nat. Acad. Sci.*, **11**, 428 (1925).
[6] Loeb, L. B., *J. Franklin Inst.*, **196**, 537 and 771 (1923).
[7] Blanc, A., *J. Physique*, **7**, 825 (1908).
[8] Loeb, L. B., *J. Franklin Inst.*, **201**, 279 (1926).
[9] Erikson, H. A., *Physic. Rev.*, **20**, 117 (1922); **24**, 502 and 622 (1924).
[10] Erikson, H. A., *Ibid.*, **26**, 465, 625 and 629 (1925).
[11] Langevin, P., *Ann. Chim. Physique* (8), **5**, 238 (1905).
[12] Hassé, H. R., *Phil. Mag.*, **1**, 139 (1926).
[13] Loeb, L. B., *J. Franklin Inst.*, **201**, 285 (1926).
[14] Loeb, L. B., *Kinetic Theory of Gases*, McGraw-Hill, New York, **1927**, Chapter XI, "Section on Recombination of Ions."

A REFINEMENT OF THE MICHELSON-MORLEY EXPERIMENT

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The investigation which is the subject of this paper was undertaken to test the conclusions recently drawn by Professor D. C. Miller from his repetition of the Michelson-Morley experiment. Professor Miller interprets his observations as indicating a motion of the sun through the ether with a velocity not less than 200 kilometers per second in a direction about right ascension 262 degrees, declination 65 degrees. He supposes that a contraction of the apparatus in the direction of motion occurs, which, however, departs from that given by the formula of Lorentz and Fitzgerald by an amount corresponding to a velocity of about 10 kilometers per second. That is, the average shift of the interference pattern, as an arm of his interferometer is turned through 90 degrees from the direction perpendicular or parallel to the direction of the inferred drift, is such as would be expected if the velocity of the apparatus were 10 kilometers per second and no contraction occurred. In his paper in *Science*, April 30, 1926, he seems to have abandoned the idea that the magnitude of the

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indicated velocity depends on the altitude above sea level at which the observations are made. Such results as his, modifying as they would the fundamental physical concepts, require the fullest experimental confirmation.

On the classical ether hypothesis the well-known theory of the experiment predicts a shift of the interference pattern proportional to the lengths of the paths traversed by the interfering beams. In order to make a small velocity observable, Professor Miller used a large interferometer in which the light path was about 65 meters long. The difficulty of maintaining sufficient freedom from air currents and temperature effects with so large an instrument can be appreciated when one considers that the shift corresponding to 10 kilometers per second could also be brought about by a change in optical length of one path of less than one part in 10^9. As slight a difference of average density of air along the two arms as would be produced by a pressure difference of 2×10^{-3} mm. of mercury or a temperature difference of 10^{-3} degrees C. would suffice to yield effects of the magnitude observed.

In the present work the light paths were reduced to about 4 meters, and the required sensitiveness obtained by an arrangement capable of detecting a very slight displacement of the interference pattern. The whole optical system was enclosed in a sealed metal case containing helium at atmospheric pressure. Because of its small size the apparatus could be effectively insulated, and circulation and variations in density of the gas in the light paths nearly eliminated. Furthermore, since the value of $\mu - 1$ for helium is only about one-tenth that for air at the same pressure it will be seen that the disturbing effects of changes in density of the gas correspond to those in air at only a tenth of an atmosphere of pressure. Actually it was found that any wavering of the interference pattern was imperceptible, and when temperature equilibrium had been reached there was no steady shift.

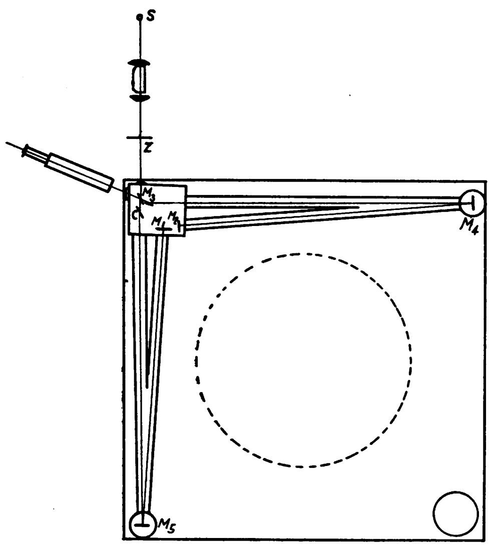

FIGURE 1

The plan of the apparatus is sketched in figure 1. The optical parts are mounted on a marble slab 122 cm. square by 10.5 cm. thick, which rests on an annular float in a pan of mercury 77 cm. in diameter. This is simply a reduced copy of Michelson's original mounting. The mirrors M_1, M_4 and M_5 are fixed in position; such adjustments of the compensating plate C and mirror M_2 as are necessary after the cover is in place can be made from the observer's position at the telescope. The green light λ 5461 is separated by the lens and prism system from the radiation of a small mercury arc lamp S attached to the slab, and passed through a small hole in the screen Z. The pencils of light are carefully limited by screens and by focusing in order to prevent stray light from reaching the eye and thereby reducing its sensitiveness. Adjustments are made so that broad fringes are formed at the surface of M_1 and M_2, on which the telescope is focused. Final adjustments are made by rotating the compensating plate C by means of a fine differential screw, and by placing small weights near the corner of the slab; under proper conditions a 5 g. weight deflects the heavy slab just perceptibly. The adjusting screws are manipulated by means of spindles passing through short flexible tubes in such a way as to be freely rotatable but air-tight. After the mirrors are given preliminary alignment, the cover is carefully lowered into place, sealed to the slab, and then filled with helium.

Schematically the arrangement of the interferometer is shown in figure 2. A beam of practically plane-parallel, homogeneous light, plane polarized so that its electric vector lies in the plane of the paper, moves to the right and falls on the mirror M_3 at the polarizing angle for the given wavelength. At the upper face the beam is split by a thin platinum film into two parts of nearly equal intensity, one passing to the mirror M_1 and the other to M_2. From there they are reflected back to M_3 where they recombine and pass to the eye through a telescope focused on M_1 and M_2. Two purposes are accomplished by the use of plane polarized light; first, the non-interfering rays indicated by the dotted lines, which would be produced with natural light, are completely eliminated, and second, the recombining beams can be adjusted to perfect equality of intensity by varying the relative reflecting powers of M_1 and M_2. Because there are two more glass-air interfaces to be traversed by the upper beam than the lower, it is impossible to equalize both components of natural light in this way.

The high sensibility necessary because of the short paths is secured chiefly by the simple device of raising one half of the surface of mirror M a small fraction of a wave-length above the other, the dividing line between the two levels being straight and as sharp as possible. The mirror used was made by covering part of a plane plate with a flat sharp-edged microscope cover-glass and applying the extra thickness by cathode deposition of platinum; thereafter giving the whole plate a fully-reflecting coat. The writer ran across the suggestion of using such a divided mirror in interferometry some years ago, but is unaware to whom the credit for it belongs.

The theory of the arrangement is as follows: The interference phenomena will be the same as if the mirror M_2 were replaced by its image in M_3. Under the conditions of the experiment, where the paths are nearly equal, M_1 is perpendicular to the beam incident on it, and the reflected beams are brought nearly to parallelism, the image of M_2 will be nearly parallel and coincident with the face of M_1. Elementary theory shows then that the resulting interference pattern will practically coincide with M_1. It would needlessly complicate this discussion to develop the general theory of interference for all inclinations of the mirrors; the experimentally realized case of near parallelism alone is necessary.

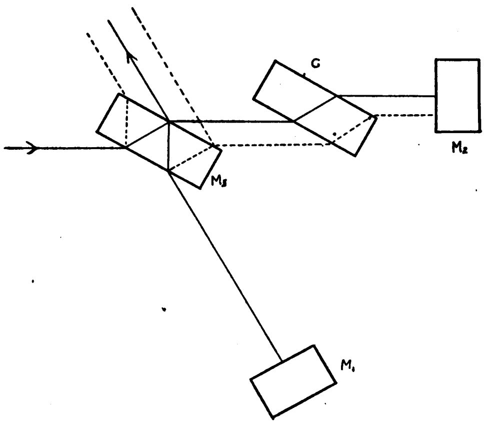

FIGURE 2

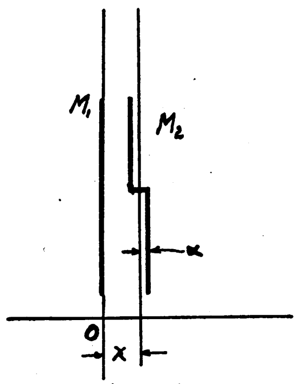

FIGURE 3

Let figure 3 represent a greatly exaggerated cross-section of M_1 and the image of M_2, normal to their planes and to the dividing line in M_2. . .M_1 lies in the plane $x = 0$, and the levels of M_2 are at equal distances on opposite sides of a parallel plane at the distance x from M_1. Let a monochromatic wave, in which the displacement is given by

$$\xi = a \cos \omega \left((t + \epsilon - \frac{x}{c} \right),$$

fall on M_1 and M_2 from the left. At the surface of M_1 the displacement in the reflected wave is then given by

$$\xi_1 = a \cos \omega \ (t + \epsilon)$$

if we ignore the loss through imperfect reflection. The displacement in the plane of M_1 in the wave reflected from the upper part of M_2 is

$$\xi_2 = a \cos \omega \left[t + \epsilon - \frac{2(x - \alpha)}{c} \right].$$

The square of the resultant displacement is then

$$(\xi_1 + \xi_2)^2 = a^2 \left\{ \cos \omega\,(t + \epsilon) + \cos \omega \left[t + \epsilon - \frac{2(x - \alpha)}{c} \right] \right\}.$$

This can be reduced to the form

$$2a^2 \left[1 + \cos \frac{2\omega}{c} (x - \alpha) \right] \cos^2 \omega\,(t - \delta).$$

Similarly the square of the resultant displacement in the interfering beams below the dividing line is found to be

$$2a^2 \left[1 + \cos \frac{2\omega}{c} (x + \alpha) \right] \cos^2 \omega\,(t - \delta).$$

The intensities, being proportional to the squares of the amplitudes, can be represented by

$$I_1 = ka^2 \left[1 + \cos \frac{2\omega}{c} (x - \alpha) \right]$$

and

$$I_2 = ka^2 \left[1 + \cos \frac{2\omega}{c} (x + \alpha) \right].$$

Now $\omega = 2\pi\nu$ where ν = frequency of the light. Hence $\frac{\omega}{c} = \frac{2\pi}{\lambda}$.

$$\therefore I_1 = ka^2 \left[1 + \cos \frac{4\pi}{\lambda} (x - \alpha) \right]$$

and

$$I_2 = ka^2 \left[1 + \cos \frac{4\pi}{\lambda} (x + \alpha) \right].$$

For values of $x = \frac{n\lambda}{4}$, where n is an integer,

$$I_1 = ka^2 \left(1 \pm \cos \frac{4\pi\alpha}{\lambda} \right)$$

the sign being positive for even values of n, and negative for odd values. The same expression holds for I_2; hence under these conditions

$$I_1 = I_2.$$

To the observer, then, the field of view is equally intense on both sides of the dividing line when $x = \frac{n\lambda}{4}$.

We have now to determine the least change in x from this value which will produce a perceptible difference in illumination in the two sides of the field. If x is given the variation δx while α is kept constant, the difference in intensity will be

$$\delta I = \left(\frac{\partial I_1}{\partial x} - \frac{\partial I_2}{\partial x}\right)\delta x.$$

Now

$$\frac{\partial I_1}{\partial x} = -\frac{4\pi k a^2}{\lambda}\sin\frac{4\pi}{\lambda}(x-\alpha)$$

$$= \pm\frac{4\pi k a^2}{\lambda}\sin\frac{4\pi\alpha}{\lambda}.$$

Similarly

$$\frac{\partial I_2}{\partial x} = \pm\frac{4\pi k a^2}{\lambda}\sin\frac{4\pi\alpha}{\lambda}.$$

$$\therefore\ \delta I = \pm\left[\frac{8\pi k a^2}{\lambda}\sin\frac{4\pi\alpha}{\lambda}\right]\delta x,$$

the sign being of no importance.

The perceptibility of the variation is determined not by δI alone, but by the ratio of δI to the total intensity, I_1 or I_2. According to the Weber-Fechner law, if δI is taken to be the least perceptible variation in intensity, the above ratio is nearly constant for a considerable range of intensities. With this meaning of δI, δx becomes the least detectable change of position of M_2.

If we have initially uniformity of illumination, we have from the equations above,

$$\frac{\delta I}{I} = \frac{8\pi}{\lambda}\delta x \frac{\sin\frac{4\pi\alpha}{\lambda}}{1 \pm \cos\frac{4\pi\alpha}{\lambda}},$$

or

$$\delta x = \frac{\lambda}{8\pi}\frac{\delta I}{I}\frac{1 \pm \cos\frac{4\pi\alpha}{\lambda}}{\sin\frac{4\pi\alpha}{\lambda}}.$$

If now $\frac{\delta I}{I}$ were a true constant we should have for the case of negative sign, which corresponds to dim illumination of the field, the sensibility of the apparatus increasing indefinitely as the factor α was made smaller. Unfortunately, however, I decreases with α and the Fechner "constant" soon diminishes rapidly. Nevertheless, the conditions of illumination and contrast here are similar to those in the half-shade polariscope, and from the theory of the Lippich instrument it appears that $\frac{\delta I}{I}$ equals about 8×10^{-3}. The lack of perfect planeness in the mirrors and of equality of intensity in the interfering beams is a further limiting factor; a little experimenting indicated that α should be not much less than 0.025λ, which was the value finally used. Substituting these values in the last equation we get

$$\delta x = 5 \times 10^{-5}\lambda$$

as the least detectable change in position of one of the mirrors. This corresponds to a change of optical length of path

$$\delta l = 2\delta x = 10^{-4}\lambda.$$

To take full advantage of the possibilities of the arrangement would have required perfect mirrors and an intenser and, therefore, hotter source of light than would have been desirable near the sensitive apparatus, as well as lengthening the interval between observations and, thereby allowing greater chance for any steady temperature shift to show itself. It was, therefore, not attempted in the experiment to go below values of δl equal to $2 \times 10^{-3}\lambda$; such variations were detectable without the least uncertainty.

The theory of the Michelson-Morley experiment is given too often to need discussion here. It turns out that as the apparatus is rotated through a right angle the interference pattern should be shifted as if by a change of length of one of the paths by a fraction of a wave-length whose maximum is

$$\delta l = \frac{l}{\lambda}\frac{v^2}{c^2}\cos^2\beta. \qquad (1)$$

Here v is the velocity of the apparatus with respect to the ether, l is the length of light path and β is angle between the plane of the apparatus and the direction of motion.

Suppose that in a set of rectangular coördinates, x_1, x_2, x_3 fixed in the ether the direction cosines of the velocity vector of the supposed ether drift are α_1, α_2, α_3, while the direction cosines of the normal to the plane of the apparatus are α_1', α_2', α_3'. Then the angle between the two lines will be given by

$$\cos\theta = \Sigma\, \alpha_i\, \alpha_i' \qquad (i = 1, 2, 3). \qquad (2)$$

Also
$$\alpha_j = \frac{x_j}{(\Sigma\, x_i^2)^{1/2}} \quad \text{and} \quad \alpha_j' = \frac{x_j'}{(\Sigma\, x_i'^2)^{1/2}}\cdot \qquad (3)$$

But with a suitable orientation of the Cartesian system, we have

$$\begin{aligned} x_1 &= r\cos\psi\cos\varphi \\ x_2 &= r\sin\psi \\ x_3 &= -r\sin\varphi\cos\psi \end{aligned}$$

where φ is the right ascension and ψ the declination of the point where the given direction cuts the celestial sphere. A similar set of equations relates the x''s with angles φ' and ψ'. Substituting these relations in equations (3) and the resulting relations in equation (2) we get

$$\cos\theta = \cos\psi\cos\psi'\cos(\varphi - \varphi') + \sin\psi\sin\psi'. \qquad (4)$$

The angle β in formula (1) is evidently the complement of θ so that δl is a maximum when $\cos\theta$ is a minimum. From (4) it is seen that this occurs when

$$\varphi' = \varphi + n\pi,$$

n being an odd integer. Substituting this value and that of $\psi' = 34° 8'$, the latitude of Pasadena, together with the values $\varphi = 262°$ and $\psi = 65°$ determined by Miller, in equation (4) the minimum of $\cos\theta$ is found to be about 0.15; consequently, the maximum of $\cos\beta$ is very nearly unity. This occurs at a sidereal time equal to $\frac{1}{15}\varphi' = \frac{262-180}{15} = 5.5$ hrs., which during the middle two weeks of September, when the present work was done corresponds to mean local solar times varying from 6.30 A.M. to 5.30 A.M.

If in equation (1) the values $l = 400$ cm., $v = 10^6$ cm. per sec., $c = 3 \times 10^{10}$ cm. per sec., $\lambda = 5.46 \times 10^{-5}$ cm., and $\cos\beta = 1$ are substituted, we find for the maximum shift to be expected with this apparatus an amount corresponding to a change in one of the paths

$$\delta l = 8 \times 10^{-3} \text{ wave-lengths},$$

which is four times the least amount detectable.

The experiment was performed in a constant-temperature room in the Norman Bridge Laboratory at various times of day, but oftenest at the time when Miller's conclusions require the greatest effect. The sensitiveness of the eye was tested for each trial by the placing or removal of a small weight on the slab before and after rotating it. There being no fluctuations in the field of view, it was unnecessary to take the average of a number of readings. As has been shown, a shift as small as one-fourth that corresponding to Miller's would be perceived. The result was perfectly definite. There was no sign of a shift depending on the orientation.

Because an ether drift might conceivably depend on altitude, the experiment was repeated at the Mount Wilson Observatory, in the 100-inch telescope building. Here again the effect was null.

It is intended to make a systematic search for a possible drift in other directions when the apparatus has been slightly modified to increase its sensitiveness and facilitate the numerous observations that will be necessary.

The writer is under great obligation to Dr. R. A. Millikan, whose continued interest has made this investigation possible.

ON THE EVALUATION OF CERTAIN INTEGRALS IMPORTANT IN THE THEORY OF QUANTA

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1. It is known[1] that the matrix of the hydrogen atom, determining the intensities of the hydrogen series lines and of their fine structure components, essentially depends on the integral

$$I = \int_0^\infty r^3 \chi\chi' dr, \qquad (1)$$

where the functions χ and χ' are solutions of the Schroedinger equation[2]

$$\frac{d^2\chi}{dr^2} + \frac{2}{r}\frac{d\chi}{dr} + \left(\frac{2\mu E}{K^2} + \frac{2\mu e^2}{K^2 r} - \frac{(k-1)k}{r^2}\right)\chi = 0. \qquad (2)$$

The symbols μ, e stand for the mass and the charge of the electron; $K = h/2\pi$ (h Planck's constant). The energy E and the integer k have different values in χ and χ'. These functions are supposed to be finite for $r = 0$ and to vanish for $r = \infty$.

Since in the case of elliptic orbits the functions χ, χ' turn out to be polynomials multiplied by an exponential, the direct evaluation term by term is possible on principle. The numerical computations involved are, however, so lengthy as to make this method almost prohibitive in practice. We give, therefore, in this note a reduction of the integral (1) to a simple and convenient expression. Such a reduction is quite indispensable in the case of hyperbolic orbits. Moreover, the procedure applied has an interest beyond the special case of the Kepler motions, since quite analogous expressions occur in other problems of the quantum theory. In fact, the same method has been used by the author for reducing the intensity expressions of the components in the Stark effect.[3] Only the simple closed Kepler motion, neglecting relativity effect, and spin of the electron, will be considered in the following sections.