method. Since both approaches lead to the same result, the latter procedure will be followed here.

Using $F_{\mu\nu}'=G_{\mu\alpha}G_{\nu\beta}F_{\alpha\beta}$, where $F_{\mu\nu}$ is the electromagnetic field-strength tensor, one obtains

$$E_x'=E_x, \quad (34a)$$

$$E_y'=(e^{-\delta/2}/\sqrt{\alpha})(\Gamma^{1/2}E_y-\beta B_z), \quad (34b)$$

$$E_z'=(e^{-\delta/2}/\sqrt{\alpha})(\Gamma^{1/2}E_z+\beta B_y), \quad (34c)$$

and

$$B_x'=B_x, \quad (35a)$$

$$B_y'=(e^{\delta/2}/\sqrt{\alpha})(\Gamma^{1/2}B_y+\beta E_z), \quad (35b)$$

$$B_z'=(e^{\delta/2}/\sqrt{\alpha})(\Gamma^{1/2}B_z-\beta E_y). \quad (35c)$$

The transformed fields for the antiparticle can be obtained in a similar manner. Because of the difference between the matrices $B_{\rm I}$ and $B_{\rm II}$, a different result will be found. This shows clearly that, according to this generalization of special relativity, particles and their antiparticles have different transformation properties.

Notice that, depending upon the sign of δ, the transformed electric and magnetic fields will be damped or augmented by the factor $e^{\mp\delta/2}$. This prediction could perhaps be checked by using Eqs. (34a), (34b), (34c) and (35a), (35b), (35c) in the consideration of very dense and highly energetic plasmas.

VIII. CONCLUSION

Although the theory of relativistic interactions presented here lies outside the realm of current research, it appears to be a reasonable generalization of special relativity. Not only are the predictions of special and general relativity included as special cases, but also a host of new effects are predicted, all of which seem to be within range of present experimental techniques. From a theoretical point of view, the theory suggests that (a) classes of elementary particles possess distinct space-time transformations, (b) the existence of matter and antimatter is a relativistic effect independent of the quantum theory, and (c) for strong interactions new forms of the Dirac and Klein-Gordon equations should be used.

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Rotating Masses and Their Effect on Inertial Frames*

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An analysis is given of the stress-energy tensor and geometry produced by slowly rotating bodies. The geometrized mass GM/c^2 of the body is allowed to be comparable to its radius. The geometry is treated as a perturbation of the Schwarzschild geometry, which leads to considerable simplification of Einstein's equations. The rotation of the intertial frame induced by a rotating massive shell is calculated and discussed with particular attention to two limiting cases: (1) For small masses it reduces to Thirring's well-known result; (2) for large masses, whose Schwarzschild radius approaches the shell radius, the *induced rotation approaches the rotation of the shell.* These and the corresponding results for an expanding and recollapsing dust cloud are examined for their consistency with particular interpretations of Mach's principle. The analytic extension of the rotating exterior metric is a completely source-free rotating solution. It describes a slowly rotating, expanding, and recontracting Einstein-Rosen bridge which can be taken as a geometrodynamic model for a slowly rotating body.

I. INTRODUCTION

IN 1686, Newton[1] published his famous discussion of inertial forces on a fluid contained in a rotating vessel. This discussion was critically re-examined by Mach[2] in 1883 in an attempt to understand better how inertial forces arise. He suggested that the shape of the water-surface may depend on the rotation of the vessel "if the sides of the vessel increased in thickness and mass till they were ultimately several leagues thick." A calculation of such effects became possible when Einstein[3] formulated his general theory of relativity in 1916, and was carried out in 1918 by Thirring.[4] Using the weak-field approximation to Einstein's equations, Thirring found that a slowly rotating mass shell drags along the inertial frames within it. Due to

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† National Science Foundation Predoctoral Fellow.

[1] I. Newton, *Mathematical Principles* (University of California Press, Berkeley, California, 1960), p. 10.

[2] E. Mach, *The Science of Mechanics* (Open Court Publishing Company, La Salle, Indiana, 1902), p. 232.

[3] A. Einstein, Ann. Phys. 49, 769 (1916).

[4] H. Thirring, Physik. Z. 19, 33 (1918); 22, 29 (1921).

the approximations used, Thirring's result is valid only when the induced rotation is small compared to the rotation rate of the shell. Since that time various authors[5] have stressed the importance of obtaining a strong-field solution, because these induced rotation effects can be viewed as a manifestation of Mach's principle in general relativity.

Today, it is still a matter of controversy how to give a precise formulation of Mach's principle, and whether general relativity includes Mach's principle, or needs to be supplemented by boundary conditions, or must be modified in order to be consistent with this principle.[5] However, there is general agreement that the dragging along of inertial frames by rotating masses is a Machian effect. In particular, for mass shells comprising more nearly *all* the matter in the universe than those treated by Thirring, Mach's principle suggests that the inertial properties of space inside the shell no longer depend on the inertial frames at infinity, but are completely determined by the shell itself.

In this paper we study the effect on inertial frames of arbitrarily large masses rotating slowly. Instead of flat space, a static metric ("base metric") is used as the lowest order term of an approximation expansion in the angular velocity of rotation ω_s. A general perturbation away from a static metric would give rise to various modes of gravitational radiation,[6] and therefore show secular effects in higher orders. Here we confine attention to perturbations of sufficiently high symmetry that the resulting metric is stationary. Such perturbations exist only for very special base metrics, such as the spherically symmetric ones considered in this paper, and axially symmetric metrics in general.

Our model of Newton's vessel is a spherical shell of matter. To first order in ω_s the space inside the shell is flat, so that extended inertial frames can be defined there, and compared with the inertial frames at infinity. By matching this interior solution to an exterior solution, we will obtain the induced rotation rate Ω of the inertial frames.

The exterior first-order solution itself is of interest, because it can be analytically continued almost as far as the Schwarzschild solution can be continued. The result describes a rotating Einstein-Rosen bridge, a solution of the source-free equations exhibiting mass *and* angular momentum. Just as the analytic extension of the Schwarzschild solution is no longer static but shows an expanding and recollapsing throat, the corresponding rotating solution shows a time dependence. The perturbation increases to infinity as the throat collapses, and therefore is a valid approximation only during the expanded stage. Thus the present analysis cannot answer the question whether rotation will stop collapse.

II. NONROTATING BASE METRIC

The base metric is the Schwarzschild solution, written here in isotropic spherical coordinates:

$$ds^2=\psi^4(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2)-V^2dt^2. \tag{1}$$

In the exterior, source-free region, ψ and V have the form

$$\begin{aligned}\psi&=1+\alpha/r,\\ V&=(r-\alpha)/(r+\alpha),\end{aligned} \tag{2}$$

where $\alpha=\frac{1}{2}m$. In order to facilitate physical interpretation and to expedite the calculations, the orthonormal Cartan frame components of *all* tensors will be used. A convenient set of these frames is

$$\begin{aligned}\omega^0&=Vdt,\\ \omega^1&=\psi^2dr,\\ \omega^2&=r\psi^2d\theta,\\ \omega^3&=r\psi^2\sin\theta d\phi.\end{aligned} \tag{3}$$

For any spherically symmetric shell of radius r_0, and thickness small compared to r_0, the interior is flat space; the values of ψ and V are determined by continuity and the field equation

$$G^{00}=-4\nabla^2\psi/\psi^5=8\pi T^{00}. \tag{4}$$

One finds

$$\alpha=2\pi\int_0^\infty T^{00}r^2\psi^5dr, \tag{5a}$$

$$\left.\begin{aligned}\psi&=1+\alpha/r_0\\ V&=(r_0-\alpha)/(r_0+\alpha)\end{aligned}\right\}\text{ for } r<r_0. \tag{5b}$$

III. ROTATING METRIC

Consider the perturbation of the isotropic Schwarzschild metric, for a thin massive shell (suggested by Thirring's weak field result),

$$ds^2=\psi^4[dr^2+r^2d\theta^2+r^2\sin^2\theta(d\phi-\Omega(r)dt)^2]-V^2dt^2. \tag{6}$$

A convenient orthonormal set of Cartan frames is

$$\begin{aligned}\omega^0&=Vdt,\\ \omega^1&=\psi^2dr,\\ \omega^2&=r\psi^2d\theta,\\ \omega^3&=r\psi^2\sin\theta(d\phi-\Omega(r)dt).\end{aligned} \tag{7}$$

[5] See, e.g., R. H. Dicke, in *Relativity Groups and Topology* (Gordan and Breach Science Publishers, Inc., New York, 1964); J. A. Wheeler, *ibid.*

[6] For an example of a general perturbation on a nonflat base metric, see D. Brill and J. Hartle, Phys. Rev. 135, B271 (1964). R. Bach, Z. Math. 13, 119 (1922) performed a first- and second-order perturbation expansion with the Schwarzschild metric as the base metric. His perturbations correspond to stationary rotation in the exterior region, and the form of the solution in first order agrees with the results of the present paper. However, Bach does not discuss the interior solution and uses Thirring's expression to connect the integration constants in the metric with the parameters of the source. Therefore, the dragging along of the inertial frame cannot be deduced from his work to any higher accuracy than from that of Thirring.

If the metric (6) is transformed to a system rotating with angular velocity Ω_0 about the Z axis, it has the form (6) with $\Omega(r)$ replaced by $\Omega(r)-\Omega_0$. Hence we can, and always will in the following, set

$$\Omega(\infty)=0. \tag{8}$$

Then $\Omega(r)$ describes a local rotation of the inertial frames with respect to infinity.

IV. STRESS-ENERGY TENSOR FOR THIN SHELL

The distribution of elastic stresses in a nonrotating shell is completely determined by the Einstein field equations. Integration of the equation

$$G^{11}=8\pi T^{11} \tag{9}$$

across the thin shell yields

$$T^{11}=0. \tag{10}$$

Integration of the equation

$$G^{22}=8\pi T^{22} \tag{11}$$

across the shell yields

$$T^{22}=\rho\alpha/2(r_0-\alpha). \tag{12}$$

Here ρ is the mass density in the rest frame of an element of the shell. Similarly we obtain

$$T^{33}=T^{22}, \tag{13}$$

since

$$G^{22}=G^{33}=\frac{(r\psi^2)_1V_1}{r\psi^6V}+\frac{1}{r\psi^4}\left[\frac{(r\psi^2)_1}{\psi^2}\right]_1+\frac{1}{V\psi^2}\left[\frac{V_1}{\psi^2}\right]_1, \tag{14}$$

where subscript 1 denotes differentiation with respect to r.

The stress-energy tensor for the same shell when it is rotating with angular velocity ω_s has the form

$$T^{\mu\nu}=\rho u^\mu u^\nu+\sum_{i,j=1}^{3} t^{ij}v_{(i)}{}^\mu v_{(j)}{}^\nu. \tag{15}$$

Here ρ is again the rest mass density, and u^μ the velocity four vector, of an element of the shell; the $v_{(i)}{}^\mu$ form a triad of orthonormal vectors spanning the hypersurface orthogonal to u^μ. $T^{\mu\nu}$ must have this form because in the rest frame of the matter the momentum density T^{i0} should vanish. The Cartan-frame components of the velocity four-vector are given by[7]

$$u^\mu=[\omega^\mu/d\tau]_{\text{along motion}}. \tag{16}$$

For a spherical shell rotating about the z axis the expression for u^μ becomes

$$u^0=(1-\sigma^2)^{-1/2},\quad u^1=u^2=0,\quad u^3=\sigma/(1-\sigma^2)^{1/2}, \tag{17}$$

where $\sigma=r\psi^2\sin\theta(\omega_s-\Omega)/V$.

For two of the vectors $v_{(i)}{}^\mu$ we choose unit vectors along the ω_1 and ω_2 directions, which are automatically orthogonal to each other and to u^μ; the remaining vector is then uniquely determined by orthonormality and the requirement that it be parallel to the third Cartan basis vector in the limit of no rotation.

$$\begin{aligned} v_{(1)}{}^\mu&=(0,1,0,0),\\ v_{(2)}{}^\mu&=(0,0,1,0);\\ v_{(3)}{}^\mu&=(\sigma,0,0,1)/(1-\sigma^2)^{1/2}. \end{aligned} \tag{18}$$

Many of the components $T^{\mu\nu}$ vanish owing to the reflection symmetry about the equatorial plane $(\theta\to\pi-\theta)$ and time-reversal symmetry $(t\to -t,\ \phi\to-\phi,\ \omega_s\to-\omega_s,\ \Omega\to-\Omega)$. We conclude that (a) the elastic stress tensor t^{ij} is diagonal, (b) T^{00} and T^{ij} are even functions of $\omega_s-\Omega$, and (c) T^{0i} are odd functions of $\omega_s-\Omega$. Equation (15) now gives, correct to first order,

$$\begin{aligned} T^{00}&=\rho,\\ T^{22}&=t^{22},\\ T^{33}&=t^{33},\\ T^{03}&=(\rho+t^{33})\sigma, \end{aligned} \tag{19}$$

and from the zeroth-order result we have[8]

$$t^{22}=t^{33}=\rho\alpha/2(r_0-\alpha)\equiv\rho\beta\quad(\text{definition of }\beta). \tag{19a}$$

---

[7] *Computation of four-velocity in Cartan frames.* Let the motion of a particle be described in parametric form, $x^\mu=x^\mu(\tau)$, where τ is the arc length along the path. The holonomic components $u_H{}^\mu$ of the four-velocity are defined by

$$u_H{}^\mu=dx^\mu/d\tau. \tag{A}$$

Let e_μ be the natural basis associated with the coordinate system and let ω_ν be the orthonormal basis of the Cartan frames. The two are related by some linear transformation

$$e_\mu=\omega_\nu\alpha_\mu{}^\nu(x). \tag{B}$$

The four-vector **u** can now be expressed in two ways:

$$\mathbf{u}=e_\mu u_H{}^\mu=e_\mu dx^\mu/d\tau=\omega_\nu\alpha_\mu{}^\nu(x)dx^\mu/d\tau=\omega_\nu u^\nu. \tag{C}$$

Comparison of the two expressions yields the Cartan frame components of the velocity four-vector,

$$u^\nu=\alpha_\mu{}^\nu(x)dx^\mu/d\tau. \tag{D}$$

Since

$$\alpha_\mu{}^\nu(x)dx^\mu=\omega^\nu \tag{E}$$

we can write Eq. (D) symbolically as

$$u^\nu=[\omega^\nu/d\tau]_{\text{along motion}}. \tag{F}$$

Here ω^ν does not have its usual *meaning* of a differential form (viz., dual basis to the ω_μ), but is the same *expression* in terms of the coordinate differentials. (Alternately, define for each point four functions $f^{(\nu)}$ such that *locally* $df^{(\nu)}=\omega^\nu$; then u^ν is the *derivative* $df/d\tau$ at that point.) Thus we find, for example,

$$\begin{aligned} u^3=\omega^3/d\tau&=r\psi^2\sin\theta(d\phi-\Omega dt)/[V^2dt^2-r^2\psi^4\sin^2\theta(d\phi-\Omega dt)^2]^{1/2}\\ &=r\psi^2\sin\theta(\omega_s-\Omega)dt/[1-r^2\psi^4\sin^2\theta(\omega_s-\Omega)^2/V^2]^{1/2}Vdt, \end{aligned}$$

where we have put $d\phi/dt=\omega_s$.

[8] This result shows that Thirring (Ref. 4) was correct in neglecting the elastic stress T^{ij} in the shell to first order in ω_s for slow rotation and small masses producing weak gravitational fields. However, for large masses, t^{ij} must be large in order to prevent collapse of the shell, and makes the largest contribution to the angular momentum of the shell. Also see L. Bass and F. A. E. Pirani, Phil. Mag. 46, 850 (1955).

V. THE INDUCED ROTATION

The Einstein equations, evaluated to first order in ω_s, for metric (6) and source tensor (19) reduce to five field equations, four of which are identical with those for the nonrotating shell. The one remaining field equation

$$8\pi T^{03}=G^{03}=-[(r^2\psi^2\Omega_1/2V)_1+(r\psi^2)_1 r\Omega_1/V]\times\sin\theta/r\psi^4 \quad (20)$$

determines $\Omega(r)$: for $r\neq r_0$, ρ vanishes and the equation has the first integral

$$\Omega_1=KV\psi^2/(r\psi^2)^4, \quad (21)$$

where K is an integration constant. Another integration, using the boundary condition (8), $\Omega(\infty)=0$, yields

$$\Omega=-K/3(r\psi^2)^3 \quad r>r_0. \quad (22)$$

In the interior of the shell, the only regular solution is constant rotation,

$$\Omega=K' \quad r<r_0. \quad (23)$$

Continuity across the shell requires

$$K'=-K/3(r_0\psi_0^2)^3. \quad (24)$$

The subscript zero denotes that the quantity is evaluated at the shell where $r=r_0$. Integrating Eq. (20) across the thin shell yields

$$-\int_-^+[(r^2\psi^2\Omega_1/2V)_1+(r\psi^2)_1 r\Omega_1/V]dr =8\pi\int_-^+[\rho(1+\beta)r^2\psi^6(\omega_s-\Omega)/V]dr. \quad (25)$$

Here the limits of the region containing the mass are denoted by $-$ and $+$.

The second term of the left integral (25) goes to zero, and we find

$$K=-4m(1+\beta_0)(r_0\psi_0^2)^2\psi_0(\omega_s-K')/V_0. \quad (26)$$

Thus Ω is completely determined in the exterior and interior of the shell,

$$\Omega=\begin{cases}\dfrac{(r_0\psi_0^2/r\psi^2)^3\omega_s}{1+[3(r_0-\alpha)/4m(1+\beta_0)]}, & r>r_0.\\ \omega_s/(1+[3(r_0-\alpha)/4m(1+\beta_0)]), & r<r_0.\end{cases} \quad (27)$$

For small α the result for the interior reduces to

$$\Omega=\omega_s(4m/3r_0), \quad (28)$$

in agreement with Thirring's well-known result.[4]

V. DISCUSSION

Figure 1 shows a graph of $\Omega(r)/\omega_s$ for shells of various radii r_0 but identical total shell mass. As the shell mass α increases compared to the shell radius r_0, the dragging effect of the rotating shell on the inertial frames increases until finally, as α approaches r_0, the rotation rate Ω of the inertial frame approaches the shell rotation rate ω_s. In other words, in this limit the inertial properties of space inside the shell no longer depend on the inertial frames at infinity, but are completely determined by the shell itself. (Of course, the behavior of the interior is not completely independent of all the features of the asymptotic region, since the boundary condition of asymptotic flatness enters in an essential way into the derivation).

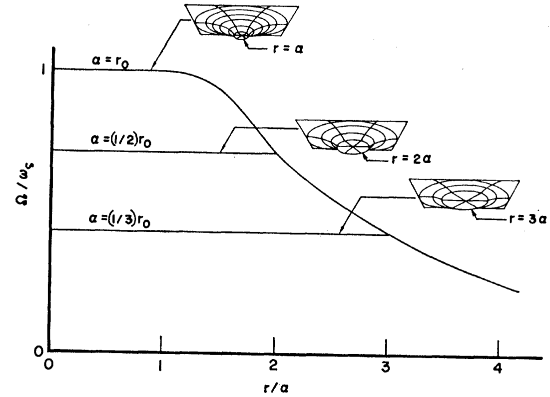

FIG. 1. Ratio of induced rotation Ω of inertial frames to rotation ω_s of inducing shell as a function of distance r from the center. Curves are plotted for shells of various radii r_0 but the same total mass. The curves coincide in the exterior region due to this normalization of total mass. [*Note added in proof.* These curves coincide only if the ordinate is multiplied by a suitable factor, which is different for each curve. The value 1 shown on the ordinate is correct only for the curve labeled $\alpha=r_0$. The correct scale for the other curves can be obtained from Eq. (27) and the definition of β_0, Eq. (19a). A similar comment applies to Fig. 2.] In the interior, "perfect dragging" ($\Omega/\omega_s\rightarrow 1$) is obtained in the limit shell radius $\rightarrow$ gravitational radius corresponding to shell mass. The physical shape of the curved space $t=$const is shown for each curve via an imbedding of the two-dimensional analog in flat three-space.

A shell of matter of radius equal to its Schwarzschild radius has often been taken as an idealized cosmological model of our universe. Our result shows that in such a model there cannot be a rotation of the local inertial frame in the center relative to the large masses in the universe. In this sense our result explains why the "fixed stars" are indeed fixed in our inertial frame, and in this sense the result is consistent with Mach's principle.

A more realistic model of a dynamic universe within the class of asymptotically flat spaces would be an expanding and recollapsing ball of dust. The simplest features of a slowly rotating dustball are found at the time of maximum expansion (time of "momentary stationarity"). Since the field equations relevant to first order in the rotation rate ω_s, $G^{0i}=8\pi T^{0i}$, are initial value equations,[9] the problem can be discussed without reference to the later time development. The results

[9] A. Lichnerowicz, *Théories relativistes de la gravitation et de l'electromagnetism* (Masson et Cie., Paris, 1955); Y. Fourès-Bruhat, J. Rat. Mech. Anal. **5**, 951 (1956). For a summary, see D. Brill, Nuovo Cimento Suppl. **2**, 1 (1964).

of the discussion are illustrated in Fig. 2. Again we consider dustballs of identical total mass but different radii. As the radii decrease beyond the Schwarzschild radius (as measured in isotropic coordinates), the proper volume of the dustball actually increases, and the interior geometry approaches that of a closed universe. In the limit of nearly complete closure we again find that the rotation rate of the ball and the inertial frame become identical. This result is consistent with the conjecture that Mach's principle is satisfied in a closed Friedman universe.

VI. ROTATION IN GEOMETRODYNAMICS

Kruskal's analytic extension[10] of the Schwarzschild metric represents a solution of the Einstein equations which is everywhere free from sources. As shown in Fig. 3, it has two-sheeted spacelike surfaces and shows a dynamic behavior in time, in particular, a collapse of the throat region in a finite proper time. One sheet and its time development correspond to the Schwarzschild coordinate $R=r\psi^2$ range $0<R<\infty$; the other sheet is a replica of this same geometry joined analytically to the first.

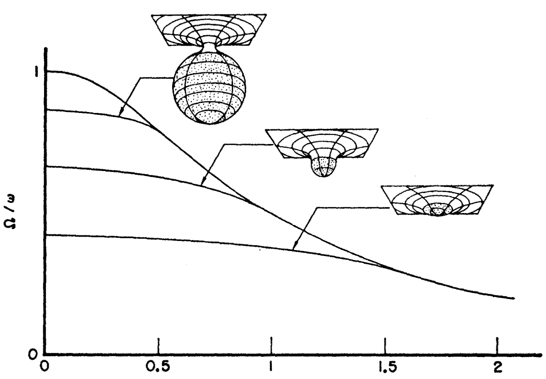

FIG. 2. Ratio of induced rotation Ω to rotation ω of expanding and recontracting ball of dust as a function of isotropic coordinate r, at the moment $t=0$ of maximum expansion. Curves are plotted for balls of various radii r_0 but same total mass as seen by an observer at infinity. In the exterior region the physics is the same as in the corresponding region of Fig. 1, but we have chosen a time scale such that $V=1$ in order to be able to discuss the transition to a closed universe. The exterior solution satisfying the boundary condition (8) is

$$\Omega=\text{const}\times(\alpha^5+5r\alpha^4+10r^2\alpha^3)/(r+\alpha)^5.$$

The equation determining Ω in the interior is obtained by substituting $V=1$, and the solution for ψ of Eq. (4) for a ball of dust, into Eq. (20). This equation is easily seen to be *homogeneous* in $\Omega-\omega$. At $r_0\to 0$, the exterior $d\Omega/dr$ goes to zero. The matching interior solution must therefore approach the unique solution with vanishing derivative, $\Omega-\omega=0$, for small r_0. For small r_0, ψ is large and the physical size of the dustball is quite different from r_0. The physical shape of the curved space $t=0$ is shown for each curve via an imbedding of the two-dimensional analog in flat three-space. The limit $r_0\to 0$ corresponds to vanishing throat radius in comparison with the size of the dustball, and in this sense is the limit of a closed universe.

[10] M. Kruskal, Phys. Rev. **119**, 1743 (1960); R. W. Fuller and J. A. Wheeler, *ibid.* **128**, 919 (1962); also see R. H. Boyer and R. W. Lindquist (to be published).

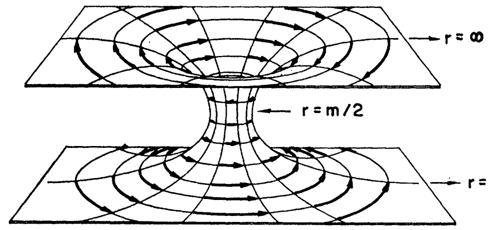

FIG. 3. Two-dimensional cross-section, $\theta=\frac{1}{2}\pi$, of curved spacelike surface $t=0$ of rotating Schwarzschild-Kruskal geometry imbedded in flat Euclidean space. The rotation can be discovered geometrically only by examining the imbedding of this surface in the four-dimensional *curved* solution of Einstein's equations. Here the quantity Ω measuring the rotation (Eq. 29) is indicated by arrows proportional to the "velocity of rotation," ΩR.

To find an analytic continuation of the corresponding rotating metric, we take the solution of Eq. (20)

$$\Omega=k[(2m)^{-3}-R^{-3}]\quad(k=K/3).\tag{29}$$

This differs from solution (22) only by the addition of a constant

$$\Omega(\infty)=k/(2m)^3.\tag{30}$$

This solution (29) can be extended into the range $0<R<\infty$ without singularity; moreover, a Kruskal-type transformation,

$$\begin{aligned}u&=e^{R/4m}[(R/2m)-1]^{1/2}\cosh(t/4m),\\ v&=e^{R/4m}[(R/2m)-1]^{1/2}\sinh(t/4m),\end{aligned}\tag{31}$$

brings the metric into the form (to first order in Ω)

$$\begin{aligned}ds^2=f^2(u,v)(du^2-dv^2)+R^2(u,v)d\Omega^2\\ +2ke^{-R/2m}(R^2+2mR+(2m)^2)\\ \times\sin^2\theta\, d\phi(udv-vdu)/mR.\end{aligned}\tag{32}$$

Finally the nonvanishing components of the Riemann tensor are (in the orthonormal frames)

$$\begin{aligned}&R^{12}{}_{12}=R^{13}{}_{13}=R^{03}{}_{03}=R^{02}{}_{02}=-2\alpha/R^3,\\ &R^{32}{}_{32}=R^{01}{}_{01}=4\alpha/R^3,\\ &-R^{02}{}_{32}=R^{01}{}_{31}=K[1-(2m/R)]^{1/2}\sin\theta/2R^4.\end{aligned}\tag{33}$$

Neither Ω, nor the Kruskal form of the metric (32), nor the Riemann tensor show any singularities at $R=2m$. Therefore the solution with rotation can be analytically extended into nearly the same region as the Schwarzschild solution. This continuation describes a slowly rotating, collapsing Einstein-Rosen bridge, a geometrodynamic model of a slowly rotating body. In the late collapse stage, when R approaches zero, Ω increases without limit. No matter how small Ω was initially, it will reach values for which the first-order approximation ceases to be applicable at the small but finite value of $R\sim k^{1/3}$. Thus our analysis does not permit us to follow the contraction of the rotating Einstein-Rosen bridge beyond this finite R value, and we cannot determine whether the rotation prevents the collapse or not.